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AN ULM-LIKE METHOD FOR INVERSE SINGULAR VALUE PROBLEMS*

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Abstract. In this paper, an Ulm-like method is proposed for solving inverse singular value problems. This method has an advantage over Newton's methods since it avoids solving approximate Jacobian equations. Under some mild assumptions, we show that the proposed method converges at least quadratically in the root sense. Our numerical tests, based on comparison with the inexact Newton method given by Bai and Xu [*Linear Algebra Appl.*, 429 (2008), pp. 527–547], demonstrate the effectiveness of the new method.

Key words. inverse singular value problem, Newton method, Ulm-like method

AMS subject classifications. 65F18, 65F10, 15A18

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1. Introduction. Inverse eigenvalue problems (IEPs) arise in various applications such as geophysics, control design, exploration and remote sensing, principal component analysis, molecular spectroscopy, particle physics, circuit theory, and applied mechanics, etc. One may refer to [1], [2], [6], [9], [10], [11], [12], [22] for the applications, mathematical theory, and algorithmic aspects of general IEPs. As a natural extension of IEPs, inverse singular value problems (ISVPs) also have a growing importance in practical applications including the optimal sequence design for direct-spread code division multiple access [20] and the construction of nonnegative and positive matrices from given singular values [14], [15].

In this paper, we consider the following ISVP.

Given $n+1$ real m-by-n matrices $A_0, A_1, \ldots, A_n$ with $m \geq n$, and n nonnegative real numbers with an order $\sigma_1^* \geq \sigma_2^* \geq \cdots \geq \sigma_n^*$, find a vector $\mathbf{c} = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ such that $\{\sigma_j^*\}_{j=1}^n$ are exactly the singular values of the matrix $A(\mathbf{c})$ defined by

$$A(\mathbf{c}) \equiv A_0 + c_1 A_1 + c_2 A_2 + \cdots + c_n A_n. \tag{1.1}$$

The ISVP was first addressed by Chu in [8], where the author gave a continuous approach and an iterative approach for solving the ISVP. In particular, the iterative approach is actually Newton's method which generalizes a numerical method proposed by Friedland, Nocedal, and Overton [12] for solving a kind of IEP.

In this paper, we propose an Ulm-like method for solving the ISVP. This is motivated by four papers [3], [5], [18], [19]. In [5], an inexact Newton-type method was given for solving the ISVP. In [3], [18], [19], Ulm-like methods (see, e.g., [21]) were presented for solving IEPs. Ulm's method is an iterative method like Newton's method, which was originally proposed for solving a nonlinear equation of $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{g}$ is a

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Fréchet-differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in another Banach space $\mathcal{Y}$. The Ulm method generates a sequence of

$$\mathbf{x}_{k+1} = \mathbf{x}_k - Q_k\mathbf{g}(\mathbf{x}_k), \quad Q_{k+1} = Q_k + (\mathcal{I} - Q_k\mathbf{g}'(\mathbf{x}_{k+1}))Q_k, \quad k = 0, 1, \ldots. \tag{1.2}$$

Here, $\mathcal{I}$ denotes the identity operator, $\mathbf{g}'(\mathbf{x}) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and $Q_k \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ $(k \geq 0)$. The Ulm method (1.2) generates a sequence $\{\mathbf{x}_k\}$ which converges to a locally unique solution $\mathbf{x}^*$ of $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ at least quadratically in the root sense. For the definition of convergence rate in the root sense, see [16, Chapter 9] or Definition 3.8 in section 3. Moreover, the method provides successive approximation Q_k to $\mathbf{g}'(\mathbf{x}^*)^{-1}$ simultaneously.

Our proposed Ulm-like method can avoid solving approximate Jacobian equations that appear in the inexact Newton-type method and thus reduce the complication caused by possible ill-conditioned Jacobian equations. Under some mild assumptions, we show that our method converges at least quadratically in the root sense. Numerical tests demonstrate that the proposed method is very effective.

Throughout the paper, we use the following notation. Let $\|\cdot\|$ be the Euclidean vector norm or its induced matrix norm, and let $\|\cdot\|_F$ denote the Frobenius norm. We use I to denote an identity matrix of appropriate size. For any $\mathbf{c} \in \mathbb{R}^n$, $\{\sigma_j(A(\mathbf{c}))\}_{j=1}^n$, $\{\mathbf{u}_j(A(\mathbf{c}))\}_{j=1}^m$, and $\{\mathbf{v}_j(A(\mathbf{c}))\}_{j=1}^n$ stand for the singular values, the left singular vectors, and the right singular vectors of $A(\mathbf{c})$, respectively. Let $\boldsymbol{\sigma}^* = (\sigma_1^*, \ldots, \sigma_n^*)^T \in \mathbb{R}^n$ and $\Sigma^* = \operatorname{diag}(\sigma_1^*, \ldots, \sigma_n^*) \in \mathbb{R}^{m\times n}$.

The paper is organized as follows. In section 2, we present the Ulm-like method for the ISVP. In section 3, we show that our Ulm-like method converges at least quadratically in the root sense. Numerical tests are reported in section 4, and some concluding remarks are given in section 5.

2. An Ulm-like method.

In this section, we propose an Ulm-like method for solving the ISVP. We note that solving the ISVP is equivalent to finding a solution to the nonlinear equation

$$\mathbf{f}(\mathbf{c}) \equiv \boldsymbol{\sigma}(A(\mathbf{c})) - \boldsymbol{\sigma}^* = \mathbf{0}, \qquad \boldsymbol{\sigma}(A(\mathbf{c})) = (\sigma_1(A(\mathbf{c})), \ldots, \sigma_n(A(\mathbf{c})))^T. \tag{2.1}$$

Thus we can use the Ulm method (1.2) to solve (2.1). In contrast to the Newton method [8] and the inexact Newton method [5], where approximate Jacobian equations need to be solved, the Ulm method (1.2) is "inversion free" and keeps away from the solution of Jacobian equations. Moreover, successive approximations Q_k to $\mathbf{g}'(\mathbf{x}^*)^{-1}$ may be useful for the subsequent convergence analysis. The Ulm method (1.2) was successfully employed in solving IEPs [3], [18], [19]. This motivates us to propose the following Ulm-like method for the ISVP.

ALGORITHM 1. THE ULM-LIKE METHOD.

I. Given $\mathbf{c}^0 \in \mathbb{R}^n$, compute the normalized left singular vectors $\{\mathbf{u}_i^0 = \mathbf{u}_i(A(\mathbf{c}^0))\}_{i=1}^m$ and the normalized right singular vectors $\{\mathbf{v}_i^0 = \mathbf{v}_i(A(\mathbf{c}^0))\}_{i=1}^n$ of $A(\mathbf{c}^0)$. Form Jacobian matrix J_0 and $\mathbf{w}^0 \in \mathbb{R}^n$ by

$$[J_0]_{ij} = (\mathbf{u}_i^0)^T A_j \mathbf{v}_i^0, \quad [\mathbf{w}^0]_i = (\mathbf{u}_i^0)^T A_0 \mathbf{v}_i^0, \quad i, j = 1, \ldots, n. \tag{2.2}$$

Let $Q_0 = J_0^{-1}$, and compute $\mathbf{c}^1$ by

$$\mathbf{c}^1 = \mathbf{c}^0 - Q_0(J_0\mathbf{c}^0 + \mathbf{w}^0 - \boldsymbol{\sigma}^*) = J_0^{-1}(\boldsymbol{\sigma}^* - \mathbf{w}^0).$$

Set $\mathbf{s}^0 \equiv (s_1^0, s_2^0, \ldots, s_n^0)^T = \boldsymbol{\sigma}^*$.

II. For $k = 1, 2, \ldots,$ until convergence, do:

(i) Form the matrix $A(\mathbf{c}^k)$ by (1.1).

(ii) Form the matrix $Z_{k-1} \equiv U_{k-1}^T A(\mathbf{c}^k) V_{k-1}$.

(iii) Calculate the skew-symmetric matrices X_k and Y_k by

$$[X_k]_{ij} = 0, \qquad n+1 \le i \ne j \le m;$$

$$[X_k]_{ij} = -[X_k]_{ji} = \frac{[Z_{k-1}]_{ij}}{s_j^{k-1}}, \quad n+1 \le i \le m, \quad 1 \le j \le n;$$

$$[X_k]_{ij} = -[X_k]_{ji} = \frac{s_i^{k-1}[Z_{k-1}]_{ji} + s_j^{k-1}[Z_{k-1}]_{ij}}{(s_j^{k-1})^2 - (s_i^{k-1})^2}, \quad 1 \le i < j \le n;$$

$$[Y_k]_{ij} = -[Y_k]_{ji} = \frac{s_i^{k-1}[Z_{k-1}]_{ij} + s_j^{k-1}[Z_{k-1}]_{ji}}{(s_j^{k-1})^2 - (s_i^{k-1})^2}, \quad 1 \le i < j \le n.$$

(iv) Compute $U_k = [\mathbf{u}_1^k, \ldots, \mathbf{u}_m^k]$ and $V_k = [\mathbf{v}_1^k, \ldots, \mathbf{v}_n^k]$ by solving

$$\left(I + \frac{1}{2} X_k\right) U_k^T = \left(I - \frac{1}{2} X_k\right) U_{k-1}^T,$$

$$\left(I + \frac{1}{2} Y_k\right) V_k^T = \left(I - \frac{1}{2} Y_k\right) V_{k-1}^T.$$

(v) Form the approximate Jacobian matrix J_k and $\mathbf{w}^k \in \mathbb{R}^n$ by

$$[J_k]_{ij} = (\mathbf{u}_i^k)^T A_j \mathbf{v}_i^k, \quad [\mathbf{w}^k]_i = (\mathbf{u}_i^k)^T A_0 \mathbf{v}_i^k, \quad i, j = 1, \ldots, n.$$

(vi) Compute $Q_k \in \mathbb{R}^{n \times n}$ and $\mathbf{c}^{k+1}$ by

$$Q_k = Q_{k-1} + (I - Q_{k-1} J_k) Q_{k-1}, \qquad \mathbf{c}^{k+1} = \mathbf{c}^k - Q_k (J_k \mathbf{c}^k + \mathbf{w}^k - \boldsymbol{\sigma}^*).$$

Set $\mathbf{s}^k \equiv (s_1^k, s_2^k, \ldots, s_n^k)^T = \boldsymbol{\sigma}^* + \mathbf{t}^k$, where

$$\mathbf{t}^k \equiv (t_1^k, t_2^k, \ldots, t_n^k)^T = (I - J_k Q_k)(J_k \mathbf{c}^k + \mathbf{w}^k - \boldsymbol{\sigma}^*).$$

We note that, instead of solving successive approximate Jacobian equations, Algorithm 1 generates successive approximations Q_k to the inverse of Jacobian matrix $J(\mathbf{c}^*)$ (see (3.2) for the definition), provided that a solution $\mathbf{c}^*$ to the ISVP exists. We observe from Algorithm 1 that two orthogonal matrices U_k and V_k are given by

$$U_k = U_{k-1} \boldsymbol{\Phi}_k, \qquad V_k = V_{k-1} \boldsymbol{\Psi}_k, \tag{2.3}$$

where $\boldsymbol{\Phi}_k = (I + \frac{1}{2} X_k)(I - \frac{1}{2} X_k)^{-1}$ and $\boldsymbol{\Psi}_k = (I + \frac{1}{2} Y_k)(I - \frac{1}{2} Y_k)^{-1}$. For Algorithm 1, it is easy to check that the vector $\mathbf{c}^{k+1}$ and the skew-symmetric matrices X_{k+1} and Y_{k+1} are determined by

$$S_k + X_{k+1} S_k - S_k Y_{k+1} = U_k^T A(\mathbf{c}^{k+1}) V_k, \tag{2.4}$$

where $S_k = \boldsymbol{\Sigma}^* + T_k$ with $S_k = \mathrm{diag}(s_1^k, \ldots, s_n^k) \in \mathbb{R}^{m \times n}$ and $T_k = \mathrm{diag}(t_1^k, \ldots, t_n^k) \in \mathbb{R}^{m \times n}$. We should mention that the operation cost of Algorithm 1 is $O(n^3)$, almost the same as that of solving a Jacobian equation. However, computing the product of matrices is simpler than solving equations and has no unstability problem caused by ill-conditioning in solving equations. In particular, the parallel computation techniques can be applied in the product of matrices in the Ulm-like method to improve the computational efficiency. We will show that Algorithm 1 converges at least quadratically in the root sense in the next section.

3. Convergence analysis. In this section, we establish a quadratic convergence in the root sense of Algorithm 1 for the ISVP. Suppose that the given singular values $\{\sigma_j^*\}_{j=1}^n$ are all positive and distinct. As noted in [7], the ISVP may not have a solution since singular values cannot be assigned arbitrarily. Therefore, in what follows, we assume that the ISVP has a solution $\mathbf{c}^*$. Let $A(\mathbf{c}^*) = U^* \Sigma^* (V^*)^T$ be the singular value decomposition of $A(\mathbf{c}^*)$, where $U^* \in \mathcal{O}(m)$ and $V^* \in \mathcal{O}(n)$. It follows from Theorem 1.9.3 in [22] that there exists a neighborhood $\mathcal{B}(\mathbf{c}^*)$ at $\mathbf{c}^*$ such that the singular values $\sigma_j(A(\mathbf{c}))$ are all distinct and differentiable for all $\mathbf{c} \in \mathcal{B}(\mathbf{c}^*)$. For any $\mathbf{c} \in \mathcal{B}(\mathbf{c}^*)$, the function $\mathbf{f}(\mathbf{c})$ defined by (2.1) is nonlinear and continuously differentiable. For any $\mathbf{c} \in \mathcal{B}(\mathbf{c}^*)$, by using

$$\begin{gathered} \sigma_i(A(\mathbf{c})) = \mathbf{u}_i(A(\mathbf{c}))^T A(\mathbf{c}) \mathbf{v}_i(A(\mathbf{c})), \\ \mathbf{u}_i(A(\mathbf{c}))^T \mathbf{u}_i(A(\mathbf{c})) = 1, \qquad \mathbf{v}_i(A(\mathbf{c}))^T \mathbf{v}_i(A(\mathbf{c})) = 1, \end{gathered} \tag{3.1}$$

it is easy to derive that

$$\frac{\partial \sigma_i(A(\mathbf{c}))}{\partial c_j} = \mathbf{u}_i(A(\mathbf{c}))^T A_j \mathbf{v}_i(A(\mathbf{c})).$$

Then we get Jacobian matrix $J(\mathbf{c})$ of $\mathbf{f}$ at $\mathbf{c} \in \mathcal{B}(\mathbf{c}^*)$, where

$$[J(\mathbf{c})]_{ij} = \mathbf{u}_i(A(\mathbf{c}))^T A_j \mathbf{v}_i(A(\mathbf{c})), \quad 1 \le i, \quad j \le n. \tag{3.2}$$

We obtain by (3.1) and (3.2),

$$\boldsymbol{\sigma}(A(\mathbf{c})) = J(\mathbf{c})\mathbf{c} + \mathbf{w}(\mathbf{c}), \tag{3.3}$$

where

$$\boldsymbol{\sigma}(A(\mathbf{c})) = (\sigma_1(A(\mathbf{c})), \ldots, \sigma_n(A(\mathbf{c})))^T, \quad [\mathbf{w}(\mathbf{c})]_i = \mathbf{u}_i(A(\mathbf{c}))^T A_0 \mathbf{v}_i(A(\mathbf{c})), \quad 1 \le i \le n.$$

To show the convergence of Algorithm 1, in what follows, we assume that Jacobian matrix $J(\mathbf{c}^*)$ defined by (3.2) is nonsingular. The continuity of the matrix and its inverse ensures there exist a scalar $\delta_1 > 0$ and a constant C such that if

$$\max\{\|[\mathbf{u}_1, \ldots, \mathbf{u}_n] - U_1^*\|, \|[\mathbf{v}_1, \ldots, \mathbf{v}_n] - V^*\|\} \le \delta_1,$$

then the approximate Jacobian matrix $J = [\mathbf{u}_i^T A_j \mathbf{v}_i]$ is nonsingular and $\|J^{-1}\| \le C$.

Also, let the matrices $\mathbf{\Sigma}^*$, U^*, and U_k be partitioned as follows:

$$\mathbf{\Sigma}^* = \begin{bmatrix} \mathbf{\Sigma}_1^* \\ 0 \end{bmatrix}, \quad \mathbf{\Sigma}_1^* \in \mathbb{R}^{n\times n}; \quad U^* = [U_1^*, U_2^*], \quad U_1^* \in \mathbb{R}^{m\times n};$$
$$U_k = [U_1^k, U_2^k], \quad U_1^k \in \mathbb{R}^{m\times n}.$$

3.1. Preliminary results. We recall some necessary preliminary lemmas which can be found in [4], [5], [13].

LEMMA 3.1 (see [13, Corollary 8.6.2]). *Let $M, M+\Delta M \in \mathbb{R}^{m\times n}$ with $m \geq n$. For any $1 \leq k \leq n$,*

$$|\sigma_k(M+\Delta M) - \sigma_k(M)| \leq \|\Delta M\|,$$

where $\sigma_k(M)$ is the kth largest singular value of M.

LEMMA 3.2 (see [4, Lemma 2]). *For any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, we have*

$$\|A(\mathbf{x}) - A(\mathbf{y})\| \leq \zeta\|\mathbf{x} - \mathbf{y}\|,$$

where $\zeta = (\sum_{i=1}^n \|A_i\|^2)^{1/2}$.

LEMMA 3.3 (see [4, Lemma 4]). *Suppose that the given singular values $\{\sigma_j^*\}_{j=1}^n$ are all positive and distinct. Then there exist positive numbers δ_2 and κ such that, when $\|\mathbf{c}^k - \mathbf{c}^*\| \leq \delta_2$,*

$$\|[\mathbf{u}_1(A(\mathbf{c}^k)), \ldots, \mathbf{u}_n(A(\mathbf{c}^k))] - U_1^*\| \leq \kappa\|\mathbf{c}^k - \mathbf{c}^*\|,$$
$$\|[\mathbf{v}_1(A(\mathbf{c}^k)), \ldots, \mathbf{v}_n(A(\mathbf{c}^k))] - V^*\| \leq \kappa\|\mathbf{c}^k - \mathbf{c}^*\|.$$

LEMMA 3.4 (see [4, Lemma 5]). *Let $B \in \mathbb{R}^{n\times n}$ such that $\|B\| < 1$. Then $I - \frac{1}{2}B$ is nonsingular and*

$$\left\| \left(I + \frac{1}{2}B\right)\left(I - \frac{1}{2}B\right)^{-1} - (I+B) \right\| \leq \|B\|^2.$$

We can easily obtain the following result based on Lemma 6 in [4].

LEMMA 3.5. *Let $Z \in \mathbb{R}^{m\times n}$ and $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{m\times n}$ with $m \geq n$, where $\sigma_1 > \sigma_2 > \cdots > \sigma_n > 0$. Suppose that two skew-symmetric matrices $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{n\times n}$ satisfy*

$$X\mathbf{\Sigma} - \mathbf{\Sigma}Y = Z.$$

Then we have

$$\|X\| \leq \left(\frac{2n\sigma_1}{d} + \frac{1}{\sigma_n}\right)\|Z\|, \qquad \|Y\| \leq \frac{2n\sigma_1}{d}\|Z\|,$$

where $d = \min_{i\neq j}|\sigma_i^2 - \sigma_j^2|$.

3.2. Convergence results. Let

$$\sigma_0^* = 2\sigma_1^*, \qquad \sigma_{n+1}^* = \frac{1}{2}\sigma_n^*,$$
$$d^* = \min_{1\le i\ne j\le n} |(\sigma_i^*)^2 - (\sigma_j^*)^2|, \qquad \hat{d}^* = \min_{1\le j\le n+1} |\sigma_{j-1}^* - \sigma_j^*|,$$
$$\rho = \frac{6n\sigma_1^*}{d^*} + \frac{4}{3\sigma_n^*}, \qquad \mu_0 = 2(8nC\max_j\|A_j\|)^2,$$
$$\mu_1 = \sqrt{2}\rho(\zeta^2 C\sqrt{n} + \zeta), \qquad \mu_2 = 23\sigma_1^*,$$
$$\mu_3 = (2C+1)\sqrt{n}(\mu_2 + \sqrt{n}), \qquad \mu_4 = \sqrt{2}\rho[(\zeta+1)\mu_3 + 2\mu_2 + \sqrt{n}],$$
$$\mu = \max\{\mu_0 + 2, \mu_2 + \sqrt{n}, \mu_3, \mu_4\}.$$

We give the following result for the initial step of Algorithm 1.

THEOREM 3.6. *There exists a scalar $\delta_3 > 0$ such that, when $\|\mathbf{c}^0 - \mathbf{c}^*\| \le \delta_3$,*

$$\max\{\|U_1^0 - U_1^*\|, \|V_0 - V^*\|\} \le \kappa\|\mathbf{c}^0 - \mathbf{c}^*\| \le \delta_1/4,$$

$$\sqrt{\|X_1\|^2 + \|Y_1\|^2} \le \mu_1\|\mathbf{c}^0 - \mathbf{c}^*\| \le \delta_1/4.$$

Proof. Let $\|\mathbf{c}^0 - \mathbf{c}^*\| \le \delta_3$, where

$$\delta_3 = \min\left\{1, \delta_2, \frac{\delta_1}{4\kappa}, \frac{\delta_1}{4\mu_1}\right\}.$$

By Lemma 3.3, we have

$$\max\{\|U_1^0 - U_1^*\|, \|V_0 - V^*\|\} \le \kappa\|\mathbf{c}^0 - \mathbf{c}^*\| \le \delta_1/4.$$

Thus Jacobian matrix J_0 is invertible and $\|J_0^{-1}\| \le C$. By Lemmas 3.1 and 3.2, we obtain

$$\max_j |\sigma_j(A(\mathbf{c}^0)) - \sigma_j^*| \le \|A(\mathbf{c}^0) - A(\mathbf{c}^*)\| \le \zeta\|\mathbf{c}^0 - \mathbf{c}^*\|. \tag{3.4}$$

From Algorithm 1, (3.3), and (3.4), it follows that

$$\begin{aligned}\|\mathbf{c}^1 - \mathbf{c}^0\| &\le \|Q_0\|\cdot\|J_0\mathbf{c}^0 + \mathbf{w}^0 - \boldsymbol{\sigma}^*\| = \|Q_0\|\cdot\|J(\mathbf{c}^0)\mathbf{c}^0 + \mathbf{w}(\mathbf{c}^0) - \boldsymbol{\sigma}^*\| \\ &\le C\|\boldsymbol{\sigma}(A(\mathbf{c}^0)) - \boldsymbol{\sigma}^*\| \le C\sqrt{n}\max_j|\sigma_j(A(\mathbf{c}^0)) - \sigma_j^*| \\ &\le \zeta C\sqrt{n}\|\mathbf{c}^0 - \mathbf{c}^*\|. \end{aligned} \tag{3.5}$$

Let $\Sigma_0 = \mathrm{diag}[\sigma_1(A(\mathbf{c}^0)), \ldots, \sigma_n(A(\mathbf{c}^0))] \in \mathbb{R}^{m\times n}$. Then

$$U_0^T A(\mathbf{c}^0) V_0 = \Sigma_0. \tag{3.6}$$

We have by (2.4) and $S_0 = \Sigma^*$,

$$U_0^T A(\mathbf{c}^1) V_0 = S_0 + X_1 S_0 - S_0 Y_1 = \Sigma^* + X_1\Sigma^* - \Sigma^* Y_1. \tag{3.7}$$

By using (3.6) and (3.7), we obtain

$$X_1\Sigma^* - \Sigma^* Y_1 = U_0^T(A(\mathbf{c}^1) - A(\mathbf{c}^0))V_0 - (\Sigma^* - \Sigma_0). \tag{3.8}$$

By Lemmas 3.5 and 3.2, it follows from (3.4), (3.5), and (3.8) that

$$\begin{aligned}\sqrt{\|X_1\|^2 + \|Y_1\|^2} &\le \sqrt{2}\rho(\|A(\mathbf{c}^1) - A(\mathbf{c}^0)\| + \|\Sigma^* - \Sigma_0\|) \\ &\le \sqrt{2}\rho(\zeta\|\mathbf{c}^1 - \mathbf{c}^0\| + \max_j|\sigma_j(A(\mathbf{c}^0)) - \sigma_j^*|) \\ &\le \sqrt{2}\rho\left(\zeta^2 C\sqrt{n} + \zeta\right)\|\mathbf{c}^0 - \mathbf{c}^*\| \\ &= \mu_1\|\mathbf{c}^0 - \mathbf{c}^*\| \le \frac{\delta_1}{4}. \qquad \square\end{aligned}$$

Now, we offer the essential estimates of the quantities generated by Algorithm 1:

$$\|U_k^T A(\mathbf{c}^k)V_k - \Sigma^*\|, \quad \|\mathbf{c}^{k+1} - \mathbf{c}^k\|, \quad \|X_k\|, \quad \|Y_k\|, \quad \|U_{k+1} - U_k\|, \quad \|V_{k+1} - V_k\|.$$

THEOREM 3.7. *Suppose that* $\sigma_1^* > \sigma_2^* > \cdots > \sigma_n^* > 0$ *and Jacobian matrix* $J(\mathbf{c}^*)$ *is invertible. Let* $\{\mathbf{c}^k\}$, $\{Q_k\}$, $\{U_k\}$, *and* $\{V_k\}$ *be the sequences generated by Algorithm* 1. *Assume* $\psi_k \equiv \sqrt{\|X_k\|^2 + \|Y_k\|^2}$. *Then, there exists a constant* $\delta > 0$ *such that, when* $\|\mathbf{c}^0 - \mathbf{c}^*\| < \delta$, *the following inequalities hold for all* $k \ge 1$:

$$\|U_k^T A(\mathbf{c}^k)V_k - \Sigma^*\| \le \mu^{2^k-1}\psi_1^{2^k}, \tag{3.9}$$

$$\|I - J_k Q_k\| \le \mu^{2^k-1}\psi_1^{2^k}, \tag{3.10}$$

$$\|Q_k\| \le 2C, \tag{3.11}$$

$$\|\mathbf{c}^{k+1} - \mathbf{c}^k\| \le \mu^{2^k-1}\psi_1^{2^k}, \tag{3.12}$$

$$\psi_{k+1} \le \mu^{2^k-1}\psi_1^{2^k}, \tag{3.13}$$

$$\|U_{k+1} - U_k\| \le 2\mu^{2^k-1}\psi_1^{2^k}, \tag{3.14}$$

$$\|V_{k+1} - V_k\| \le 2\mu^{2^k-1}\psi_1^{2^k}. \tag{3.15}$$

Moreover, we have

$$\frac{3}{2}\sigma_1^* > s_1^k > \frac{1}{2}(\sigma_1^* + \sigma_2^*) > s_2^k > \frac{1}{2}(\sigma_2^* + \sigma_3^*) > \cdots > \frac{1}{2}(\sigma_{n-1}^* + \sigma_n^*) > s_n^k > \frac{3}{4}\sigma_n^* > 0$$

and

$$\min_{i\neq j}|(s_i^k)^2-(s_j^k)^2|\geq\frac{1}{2}d^*>0,$$

which make sure that Algorithm 1 *will not break down.*

Proof. Let $\|\mathbf{c}^0-\mathbf{c}^*\|<\delta$, where

$$\delta=\min\left\{1,\delta_3,\frac{1}{\mu_1},\frac{1}{2\mu_1\mu},\frac{d^*}{2\mu_1\mu(1+4\sigma_1^*)},\frac{\hat{d}^*}{2\mu_1\mu},\frac{\sigma_n^*}{2\mu_1\mu},\frac{\mu\sigma_n^*}{2\mu_1\sqrt{n}},\frac{3\delta_1}{16\mu\mu_1}\right\}. \tag{3.16}$$

We establish our theorem by the induction. We first consider the case of $k=1$.

By (3.16), we have $\delta\leq\delta_3$ and $\delta\leq 1/\mu_1$. Then by Theorem 3.6,

$$\psi_1=\sqrt{\|X_1\|^2+\|Y_1\|^2}<\mu_1\delta\leq\min\{1,\delta_1/4\},$$
$$\|U_1^0-U_1^*\|\leq\delta_1/4,\quad \|V_0-V^*\|\leq\delta_1/4. \tag{3.17}$$

Thus by Lemma 3.4 and (3.17),

$$\|U_1-U_0\|\leq 2\|X_1\|\leq 2\psi_1<\frac{\delta_1}{2},\qquad \|V_1-V_0\|\leq 2\|Y_1\|\leq 2\psi_1<\frac{\delta_1}{2}. \tag{3.18}$$

By using (3.17) and (3.18), we get

$$\|U_1^1-U_1^*\|\leq\|U_1^1-U_1^0\|+\|U_1^0-U_1^*\|<\frac{\delta_1}{2}+\frac{\delta_1}{4}<\delta_1,$$

and

$$\|V_1-V^*\|\leq\|V_1-V_0\|+\|V_0-V^*\|<\frac{\delta_1}{2}+\frac{\delta_1}{4}<\delta_1.$$

Hence, J_1 is invertible and $\|J_1^{-1}\|\leq C$. Furthermore, we have by (2.3) and Lemma 3.4,

$$\mathbf{\Phi}_1=I+X_1+G_1,\qquad \mathbf{\Psi}_1=I+Y_1+H_1, \tag{3.19}$$

where $\|G_1\|\leq\|X_1\|^2\leq\psi_1^2$ and $\|H_1\|\leq\|Y_1\|^2\leq\psi_1^2$. Using $U_1=U_0\mathbf{\Phi}_1$, $V_1=V_0\mathbf{\Psi}_1$, and (3.19), one can derive via simple computation,

$$U_1^TA(\mathbf{c}^1)V_1=S_0+R_1=\mathbf{\Sigma}^*+R_1, \tag{3.20}$$

where

$$\begin{aligned}R_1=&\ X_1(S_0-X_1S_0+S_0Y_1)Y_1-X_1^2S_0-S_0Y_1^2+G_1^T(S_0+X_1S_0-S_0Y_1)(I+Y_1)\\&+(I-X_1+G_1^T)(S_0+X_1S_0-S_0Y_1)H_1.\end{aligned}$$

Using (3.17), (3.19), and $S_0=\mathbf{\Sigma}^*$, we have

$$
\begin{aligned}
\|R_1\| &\le 3\sigma_1^*\|X_1\|\cdot\|Y_1\| + \sigma_1^*\|X_1\|^2 + \sigma_1^*\|Y_1\|^2 + 6\sigma_1^*\|X_1\|^2 + 9\sigma_1^*\|Y_1\|^2 \\
&\le \frac{3}{2}\sigma_1^*(\|X_1\|^2 + \|Y_1\|^2) + 10\sigma_1^*(\|X_1\|^2 + \|Y_1\|^2) \\
&= \frac{23}{2}\sigma_1^*\psi_1^2 \le \mu_2\psi_1^2 \le \mu\psi_1^2, \qquad (3.21)
\end{aligned}
$$

which verifies that (3.9) holds for $k = 1$.

Notice that

$$I - J_1Q_1 = I - 2J_1Q_0 + J_1Q_0J_1Q_0 = (I - J_1Q_0)^2.$$

Thus

$$
\begin{aligned}
\|I - J_1Q_1\| &\le (\|I - J_0Q_0\| + \|J_1 - J_0\|\cdot\|Q_0\|)^2 \\
&\le 2\|I - J_0Q_0\|^2 + 2\|J_1 - J_0\|^2\|Q_0\|^2. \qquad (3.22)
\end{aligned}
$$

In addition, for $1 \le i,\ j \le n$,

$$
\begin{aligned}
|[J_1]_{ij} - [J_0]_{ij}| &= |(\mathbf{u}_i^1)^T A_j \mathbf{v}_i^1 - (\mathbf{u}_i^0)^T A_j \mathbf{v}_i^0| \\
&= |(\mathbf{u}_i^1 - \mathbf{u}_i^0)^T A_j \mathbf{v}_i^1 - (\mathbf{u}_i^0)^T A_j(\mathbf{v}_i^0 - \mathbf{v}_i^1)| \\
&\le \|A_j\|(\|\mathbf{u}_i^1 - \mathbf{u}_i^0\| + \|\mathbf{v}_i^0 - \mathbf{v}_i^1\|) \\
&\le \|A_j\|(\|U_1 - U_0\| + \|V_1 - V_0\|). \qquad (3.23)
\end{aligned}
$$

By (3.18) and (3.23), we obtain

$$
\begin{aligned}
\|J_1 - J_0\| \le \|J_1 - J_0\|_F &\le n(\|U_1 - U_0\| + \|V_1 - V_0\|)\cdot \max_j\|A_j\| \\
&\le 4n\max_j\|A_j\|\psi_1. \qquad (3.24)
\end{aligned}
$$

By (3.16), we have $\delta \le 1/(2\mu_1\mu)$. Then, by using (3.17), (3.22), (3.24), and $Q_0 = J_0^{-1}$, we get

$$\|I - J_1Q_1\| \le 2C^2(4n\max_j\|A_j\|)^2\psi_1^2 \le \mu_0\psi_1^2 \le \mu\psi_1^2 < \mu\mu_1\delta < 1; \qquad (3.25)$$

i.e., (3.10) holds for $k = 1$.

The diagonal entries of (3.20) yield

$$J_1\mathbf{c}^1 + \mathbf{w}^1 - \boldsymbol{\sigma}^* = \mathbf{r}_1$$

with $\mathbf{r}_1$ being the diagonal vector of the matrix R_1. We therefore have

$$\|\mathbf{c}^2 - \mathbf{c}^1\| \le \|Q_1\|\cdot\|J_1\mathbf{c}^1 + \mathbf{w}^1 - \boldsymbol{\sigma}^*\| = \|Q_1\|\cdot\|\mathbf{r}_1\| \le \sqrt{n}\|Q_1\|\cdot\|R_1\|. \qquad (3.26)$$

Also by (3.25),

$$\|Q_1\| \le \|J_1^{-1}\|\cdot\|J_1Q_1\| \le \|J_1^{-1}\|(1 + \|I - J_1Q_1\|) \le 2C,$$

which shows that (3.11) holds for $k = 1$. Then, we immediately get by (3.21) and (3.26),

$$\|\mathbf{c}^2 - \mathbf{c}^1\| \le 2C\sqrt{n}\mu_2\psi_1^2 \le \mu_3\psi_1^2 \le \mu\psi_1^2. \tag{3.27}$$

This confirms that (3.12) holds for $k = 1$.

Recall from (3.16) that $\delta \le 1/(2\mu\mu_1)$ and $\delta \le \hat{d}^*/(2\mu\mu_1)$. We then have by (3.17), (3.20), (3.21), and (3.25),

$$\begin{aligned}
\|\mathbf{s}^1 - \boldsymbol{\sigma}^*\| &\le \|I - J_1Q_1\| \cdot \|J_1\mathbf{c}^1 + \mathbf{w}^1 - \boldsymbol{\sigma}^*\| \\
&\le \mu_0\psi_1^2\sqrt{n}\|R_1\| \le \sqrt{n}\mu_0\psi_1^2\mu_2\psi_1^2 \\
&\le \sqrt{n}\mu\psi_1^2\mu_2\psi_1^2 \le \frac{\sqrt{n}}{\mu}(\mu\psi_1)^2\mu_2\psi_1^2 \\
&\le \frac{\sqrt{n}}{\mu}(\mu\mu_1\delta)^2\mu_2\psi_1^2 \le \mu_2\psi_1^2 \le \mu\psi_1^2 \\
&< \mu\mu_1\delta \le \min\left\{\frac{1}{2}, \frac{1}{2}\hat{d}^*\right\}.
\end{aligned} \tag{3.28}$$

Thus one has

$$\frac{3}{2}\sigma_1^* > s_1^1 > \frac{1}{2}(\sigma_1^* + \sigma_2^*) > s_2^1 > \frac{1}{2}(\sigma_2^* + \sigma_3^*) > \cdots > \frac{1}{2}(\sigma_{n-1}^* + \sigma_n^*) > s_n^1 > \frac{3}{4}\sigma_n^* > 0.$$

By (3.16), we have $\delta \le d^*/(2\mu\mu_1(4\sigma_1^* + 1))$. Then we further derive from (3.28) that

$$\begin{aligned}
|(s_i^1)^2 - (s_j^1)^2| &\ge d^* - 4\sigma_1^*\|\mathbf{s}^1 - \boldsymbol{\sigma}^*\| - 2\|\mathbf{s}^1 - \boldsymbol{\sigma}^*\|^2 \\
&\ge d^* - (4\sigma_1^* + 1)\|\mathbf{s}^1 - \boldsymbol{\sigma}^*\| \\
&\ge d^* - (4\sigma_1^* + 1)\mu\mu_1\delta \ge \frac{1}{2}d^*
\end{aligned}$$

for $1 \le i \ne j \le n$.

Notice that $U_1^T A(\mathbf{c}^2) V_1 = S_1 + X_2S_1 - S_1Y_2$. Combining this with (3.20), we obtain

$$X_2S_1 - S_1Y_2 = U_1^T(A(\mathbf{c}^2) - A(\mathbf{c}^1))V_1 + R_1 + (\boldsymbol{\Sigma}^* - S_1).$$

By Lemmas 3.5 and 3.2, it follows from (3.21), (3.27), and (3.28) that

$$\begin{aligned}
\psi_2 = \sqrt{\|X_2\|^2 + \|Y_2\|^2} &\le \sqrt{2}\rho(\|A(\mathbf{c}^2) - A(\mathbf{c}^1)\| + \|R_1\| + \|\boldsymbol{\Sigma}^* - S_1\|) \\
&\le \sqrt{2}\rho(\zeta\|\mathbf{c}^2 - \mathbf{c}^1\| + \|R_1\| + \|\mathbf{s}^1 - \boldsymbol{\sigma}^*\|) \\
&\le \sqrt{2}\rho(\zeta\mu_3 + 2\mu_2)\psi_1^2 \le \mu_4\psi_1^2 \le \mu\psi_1^2.
\end{aligned} \tag{3.29}$$

This shows that (3.13) holds for $k = 1$. In addition, by (3.16), we have $\delta \le 1/(2\mu_1\mu)$. Then by (3.17) and (3.29),

$$\psi_2 \le \mu\psi_1^2 \le \mu\mu_1\delta\psi_1 \le \psi_1 < 1.$$

By Lemma 3.4, it follows from (3.29) that

$$\|U_2 - U_1\| \le 2\|X_2\| \le 2\mu_4\psi_1^2 \le 2\mu\psi_1^2, \qquad \|V_2 - V_1\| \le 2\|Y_2\| \le 2\mu_4\psi_1^2 \le 2\mu\psi_1^2,$$

which shows that (3.14) and (3.15) hold for $k = 1$.

Now, we consider the general case. Suppose that (3.9)–(3.15) hold for all positive integers less than or equal to $k-1$. By (3.16), we have $\delta \le 1/(2\mu_1\mu)$. Then, by the hypothesis, based on (3.17), it is easy to check that

$$\psi_k \le \mu^{2^{k-1}-1}\psi_1^{2^{k-1}} = (\mu\psi_1)^{2^{k-1}-1}\psi_1 \le (\mu\mu_1\delta)^{2^{k-1}-1}\psi_1 \le \psi_1 < 1. \tag{3.30}$$

By the hypothesis again, one has

$$\|I - J_{k-1}Q_{k-1}\| \le \mu^{2^{k-1}-1}\psi_1^{2^{k-1}} \le \psi_1 < 1 \tag{3.31}$$

and

$$\|U_{k-1}^T A(\mathbf{c}^{k-1}) V_{k-1} - \Sigma^*\| \le \mu^{2^{k-1}-1}\psi_1^{2^{k-1}}. \tag{3.32}$$

By (3.16) again, we have $\delta \le 1/(2\mu\mu_1)$ and $\delta \le \hat{d}^*/(2\mu\mu_1)$. Then combining (3.31) and (3.32) yields

$$\begin{aligned}
\|\mathbf{s}^{k-1} - \boldsymbol{\sigma}^*\| &= \|I - J_{k-1}Q_{k-1}\| \cdot \|J_{k-1}\mathbf{c}^{k-1} + \mathbf{w}^{k-1} - \boldsymbol{\sigma}^*\| \\
&\le \|I - J_{k-1}Q_{k-1}\| \cdot \sqrt{n}\|U_{k-1}^T A(\mathbf{c}^{k-1}) V_{k-1} - \Sigma^*\| \\
&\le \sqrt{n}\mu^{2^k-2}\psi_1^{2^k} \le \frac{\sqrt{n}}{\mu}(\mu\psi_1)^{2^k-1}\psi_1 \le (\mu\mu_1\delta)^{2^k-1}\psi_1 \\
&< \mu\mu_1\delta \le \min\left\{\frac{1}{2}, \frac{1}{2}\hat{d}^*\right\}.
\end{aligned} \tag{3.33}$$

Thus one has

$$\begin{aligned}
&\frac{3}{2}\sigma_1^* > s_1^{k-1} > \frac{1}{2}(\sigma_1^* + \sigma_2^*) > s_2^{k-1} > \frac{1}{2}(\sigma_2^* + \sigma_3^*) > \cdots \\
&\quad > \frac{1}{2}(\sigma_{n-1}^* + \sigma_n^*) > s_n^{k-1} > \frac{3}{4}\sigma_n^* > 0.
\end{aligned} \tag{3.34}$$

By (3.16) once again, we have $\delta \le d^*/(2\mu\mu_1(4\sigma_1^* + 1))$. We further deduce from (3.33) that

$$\begin{aligned}
|(s_i^{k-1})^2 - (s_j^{k-1})^2| &\ge d^* - 4\sigma_1^*\|\mathbf{s}^{k-1} - \boldsymbol{\sigma}^*\| - 2\|\mathbf{s}^{k-1} - \boldsymbol{\sigma}^*\|^2 \\
&\ge d^* - (4\sigma_1^* + 1)\|\mathbf{s}^{k-1} - \boldsymbol{\sigma}^*\| \\
&\ge d^* - (4\sigma_1^* + 1)\mu\mu_1\delta \ge \frac{1}{2}d^*
\end{aligned}$$

for any $1 \le i \ne j \le n$.

By (2.3) and Lemma 3.4, we obtain

$$\boldsymbol{\Phi}_k = I + X_k + G_k, \qquad \boldsymbol{\Psi}_k = I + Y_k + H_k, \tag{3.35}$$

where $\|G_k\| \le \|X_k\|^2 \le \psi_k^2$ and $\|H_k\| \le \|Y_k\|^2 \le \psi_k^2$. Using $U_k = U_{k-1}\boldsymbol{\Phi}_k$, $V_k = V_{k-1}\boldsymbol{\Psi}_k$, and (3.35), we get by simple calculation,

$$U_k^T A(\mathbf{c}^k) V_k = S_{k-1} + R_k = \Sigma^* + (S_{k-1} - \Sigma^*) + R_k, \tag{3.36}$$

where

$$\begin{aligned} R_k = {} & X_k(S_{k-1} - X_k S_{k-1} + S_{k-1} Y_k) Y_k - X_k^2 S_{k-1} - S_{k-1} Y_k^2 \\ & + G_k^T (S_{k-1} + X_k S_{k-1} - S_{k-1} Y_k)(I + Y_k) \\ & + (I - X_k + G_k^T)(S_{k-1} + X_k S_{k-1} - S_{k-1} Y_k) H_k. \end{aligned}$$

Using (3.30), (3.34), (3.35), and the hypothesis on ψ_k, we have

$$\begin{aligned} \|R_k\| & \le 3\left(\frac{3}{2}\sigma_1^*\right)\|X_k\| \cdot \|Y_k\| + \frac{3}{2}\sigma_1^*\|X_k\|^2 + \frac{3}{2}\sigma_1^*\|Y_k\|^2 \\ & \quad + 6\left(\frac{3}{2}\sigma_1^*\right)\|X_k\|^2 + 9\left(\frac{3}{2}\sigma_1^*\right)\|Y_k\|^2 \\ & \le \frac{3}{2}\left(\frac{3}{2}\sigma_1^*\right)(\|X_k\|^2 + \|Y_k\|^2) + 10\left(\frac{3}{2}\sigma_1^*\right)(\|X_k\|^2 + \|Y_k\|^2) \\ & = \frac{23}{2}\left(\frac{3}{2}\sigma_1^*\right)\psi_k^2 \le \mu_2 \psi_k^2 \le \mu_2 \mu^{2^k - 2}\psi_1^{2^k}. \end{aligned} \tag{3.37}$$

Then by (3.33), (3.36), and (3.37),

$$\begin{aligned} \|U_k^T A(\mathbf{c}^k) V_k - \Sigma^*\| & \le \|S_{k-1} - \Sigma^*\| + \|R_k\| \le \|\mathbf{s}^{k-1} - \boldsymbol{\sigma}^*\| + \|R_k\| \\ & \le (\mu_2 + \sqrt{n})\mu^{2^k - 2}\psi_1^{2^k} \le \mu^{2^k - 1}\psi_1^{2^k}. \end{aligned} \tag{3.38}$$

We can conclude that (3.9) holds for all $k \ge 1$.

By (3.16), we have $\delta \le \delta_3$ and $\delta \le 3\delta_1/(16\mu\mu_1)$. Then by Theorem 3.6, (3.18), and the hypothesis on V_k,

$$\begin{aligned} \|V_k - V_*\| & \le \sum_{j=2}^{k} \|V_j - V_{j-1}\| + \|V_1 - V_0\| + \|V_0 - V_*\| \\ & \le \sum_{j=2}^{k} 2(\mu\psi_1)^{2^{j-1}} + 2\psi_1 + \frac{\delta_1}{4} \le \frac{2(\mu\psi_1)^2}{1 - (\mu\psi_1)^2} + \frac{\delta_1}{2} + \frac{\delta_1}{4} \\ & \le \frac{4}{3}\mu\psi_1 + \frac{\delta_1}{2} + \frac{\delta_1}{4} \le \frac{\delta_1}{4} + \frac{\delta_1}{2} + \frac{\delta_1}{4} = \delta_1. \end{aligned}$$

Similarly, we can prove that $\|U_1^k - U_1^*\| \le \delta_1$. Thus $\|J_k^{-1}\| \le C$.

As the proof of (3.10) for $k = 1$, we can obtain

$$\|I - J_k Q_k\| \le 2\|I - J_{k-1} Q_{k-1}\|^2 + 2\|J_k - J_{k-1}\|^2 \|Q_{k-1}\|^2$$

and

$$\|J_k - J_{k-1}\| \le n(\|U_k - U_{k-1}\| + \|V_k - V_{k-1}\|) \cdot \max_j \|A_j\|.$$

Thus by (3.31) and the hypothesis,

$$\|I - J_kQ_k\| \le (2+\mu_0)\mu^{2^k-2}\psi_1^{2^k} \le \mu^{2^k-1}\psi_1^{2^k} \le \psi_1 < 1, \tag{3.39}$$

and hence

$$\|Q_k\| \le \|J_k^{-1}\| \cdot \|J_kQ_k\| \le \|J_k^{-1}\|(1 + \|I - J_kQ_k\|) \le 2C;$$

i.e., (3.10)–(3.11) hold for all $k \ge 1$.

Now, we have by Algorithm 1, (3.36), and (3.38),

$$\begin{aligned} \|\mathbf{c}^{k+1} - \mathbf{c}^k\| &\le \|Q_k\| \cdot \|J_k\mathbf{c}^k + \mathbf{w}^k - \sigma^*\| \le 2C\sqrt{n}\|U_k^TA(\mathbf{c}^k)V_k - \Sigma^*\| \\ &\le 2C\sqrt{n}(\mu_2 + \sqrt{n})\mu^{2^k-2}\psi_1^{2^k} \le \mu_3\mu^{2^k-2}\psi_1^{2^k} \le \mu^{2^k-1}\psi_1^{2^k}, \end{aligned} \tag{3.40}$$

and then (3.12) holds for all $k \ge 1$.

By (3.16), one has $\delta \le 1/(2\mu\mu_1)$ and $\delta \le \hat{d}^*/(2\mu\mu_1)$. Then relations (3.17), (3.36), (3.38), and (3.39) imply that

$$\begin{aligned} \|\mathbf{s}^k - \boldsymbol{\sigma}^*\| &\le \|I - J_kQ_k\| \cdot \|J_k\mathbf{c}^k + \mathbf{w}^k - \boldsymbol{\sigma}^*\| \le \sqrt{n}\|U_k^TA(\mathbf{c}^k)V_k - \Sigma^*\| \\ &\le \sqrt{n}(\mu_2 + \sqrt{n})\mu^{2^k-2}\psi_1^{2^k} \le \mu_3\mu^{2^k-2}\psi_1^{2^k} \le \mu^{2^k-1}\psi_1^{2^k} \\ &= (\mu\psi_1)^{2^k-1}\psi_1 \le (\mu\mu_1\delta)^{2^k-1}\psi_1 < \mu\mu_1\delta \le \min\left\{\frac{1}{2}, \frac{1}{2}\hat{d}^*\right\}. \end{aligned} \tag{3.41}$$

Thus one has

$$\frac{3}{2}\sigma_1^* > s_1^k > \frac{1}{2}(\sigma_1^* + \sigma_2^*) > s_2^k > \frac{1}{2}(\sigma_2^* + \sigma_3^*) > \cdots > \frac{1}{2}(\sigma_{n-1}^* + \sigma_n^*) > s_n^k > \frac{3}{4}\sigma_n^* > 0.$$

By (3.16) again, one has $\delta \le d^*/(2\mu\mu_1(4\sigma_1^* + 1))$. We further derive from (3.41) that

$$\begin{aligned} |(s_i^k)^2 - (s_j^k)^2| &\ge d^* - 4\sigma_1^*\|\mathbf{s}^k - \boldsymbol{\sigma}^*\| - 2\|\mathbf{s}^k - \boldsymbol{\sigma}^*\|^2 \\ &\ge d^* - (4\sigma_1^* + 1)\|\mathbf{s}^k - \boldsymbol{\sigma}^*\| \\ &\ge d^* - (4\sigma_1^* + 1)\mu\mu_1\|\mathbf{c}^0 - \mathbf{c}^*\| \ge \frac{1}{2}d^* \end{aligned}$$

for any $1 \le i \ne j \le n$.

Relation (3.38) implies that

$$U_k^TA(\mathbf{c}^k)V_k - \boldsymbol{\Sigma}^* = \hat{R}_k, \qquad \|\hat{R}_k\| \le (\mu_2 + \sqrt{n})\mu^{2^k-2}\psi_1^{2^k} \le \mu^{2^k-1}\psi_1^{2^k}. \tag{3.42}$$

Combining (3.42) with (2.4) gives rise to

$$X_{k+1}S_k - S_kY_{k+1} = U_k^T(A(\mathbf{c}^{k+1}) - A(\mathbf{c}^k))V_k + \hat{R}_k - (S_k - \Sigma^*),$$

and we have by Lemma 3.5 and (3.40)–(3.42),

$$
\begin{aligned}
\psi_{k+1} &= \sqrt{\|X_{k+1}\|^2 + \|Y_{k+1}\|^2} \\
&\le \sqrt{2}\rho(\|A(\mathbf{c}^{k+1}) - A(\mathbf{c}^k)\| + \|\hat{R}_k\| + \|S_k - \Sigma^*\|) \\
&\le \sqrt{2}\rho(\zeta\|\mathbf{c}^{k+1} - \mathbf{c}^k\| + \|\hat{R}_k\| + \|\mathbf{s}^k - \boldsymbol{\sigma}^*\|) \\
&\le \sqrt{2}\rho((\zeta+1)\mu_3 + \mu_2 + \sqrt{n})\mu^{2^k-2}\psi_1^{2^k} \\
&\le \mu_4\mu^{2^k-2}\psi_1^{2^k} \le \mu^{2^k-1}\psi_1^{2^k}. \qquad (3.43)
\end{aligned}
$$

This verifies that (3.13) holds for all $k \geq 1$.

By Lemma 3.4 and (3.43), we obtain

$$\|U_{k+1} - U_k\| \le 2\|X_{k+1}\| \le 2\psi_{k+1} \le 2\mu^{2^k-1}\psi_1^{2^k}$$

and

$$\|V_{k+1} - V_k\| \le 2\|Y_{k+1}\| \le 2\psi_{k+1} \le 2\mu^{2^k-1}\psi_1^{2^k}.$$

Therefore, the estimates (3.14) and (3.15) hold for all $k \geq 1$. ☐

Finally, we show that our method converges at least quadratically in the root sense. We first recall the definition of convergence rate in the root sense [16, Chapter 9].

DEFINITION 3.8. *Let $\{\mathbf{x}^k\}$ be a sequence to a limit $\mathbf{x}^*$. Then the numbers*

$$R_p\{\mathbf{x}^k\} = \begin{cases} \limsup\limits_{k\to\infty} \|\mathbf{x}^k - \mathbf{x}^*\|^{1/k} & \text{if } p = 1, \\ \limsup\limits_{k\to\infty} \|\mathbf{x}^k - \mathbf{x}^*\|^{1/p^k} & \text{if } p > 1, \end{cases}.$$

are the convergence factors in the root sense of $\{\mathbf{x}^k\}$. The quantity

$$O_R(\mathbf{x}^*) = \begin{cases} \infty, & \text{if } R_p\{\mathbf{x}^k\} = 0 \, \forall p \in [1,\infty), \\ \inf\{p \in [1,\infty) | R_p\{\mathbf{x}^k\} = 1\}, & \text{otherwise}, \end{cases}$$

is called the convergence rate in the root sense of $\{\mathbf{x}^k\}$ at $\mathbf{x}^*$.

The following theorem gives the main convergence result whose proof is similar to that of Theorems 2 and 3 in [4] (see also [5, Theorems 4.10 and 4.12]), and therefore we omit it.

THEOREM 3.9. *Suppose that $\sigma_1^* > \sigma_2^* > \cdots > \sigma_n^* > 0$ and Jacobian matrix $J(\mathbf{c}^*)$ is invertible. Then there exists a constant $\epsilon > 0$ such that, when $\|\mathbf{c}^0 - \mathbf{c}^*\| \le \epsilon$, the sequences $\{\mathbf{c}^k\}$, $\{X_k\}$, $\{Y_k\}$, $\{U_k\}$, $\{V_k\}$, and $\{U_k^T A(\mathbf{c}^k) V_k\}$ generated by Algorithm 1 converge at least quadratically in the root sense.*

4. Numerical tests. In this section, we report some numerical tests to illustrate the effectiveness of our proposed method. We compare the numerical performance of Algorithm 1 with that of the inexact Newton method in [5]. All the tests were implemented in MATLAB 7.0 on an Intel Pentium R PC with 3.00 GHz CPU.

For the completeness of our presentation, we recall the inexact Newton method [5] as follows.

Algorithm 2. The inexact Newton method.

I. Given $\gamma \in (1, 2]$ and $\mathbf{c}^0 \in \mathbb{R}^n$, compute the singular values $\{\sigma_i^0 = \sigma_i(A(\mathbf{c}^0))\}_{i=1}^n$, the normalized left singular vectors $\{\mathbf{u}_j^0 = \mathbf{u}_i(A(\mathbf{c}^0))\}_{i=1}^m$, and the normalized right singular vectors $\{\mathbf{v}_i^0 = \mathbf{v}_i(A(\mathbf{c}^0))\}_{i=1}^n$ of $A(\mathbf{c}^0)$. Form Jacobian matrix J_0 and $\mathbf{w}^0 \in \mathbb{R}^n$ by (2.2) and apply an iterative method (e.g., the QMR method [17]) to solve Jacobian equation

$$J_0\mathbf{c}^1 = \boldsymbol{\sigma}^* - \mathbf{w}^0$$

such that

$$\|J_0\mathbf{c}^1 - \boldsymbol{\sigma}^* + \mathbf{w}^0\| \leq r^0\|\boldsymbol{\sigma}^* - \mathbf{w}^0\|,$$

where $r^0 = \|\boldsymbol{\sigma}^0 - \boldsymbol{\sigma}^*\|^\gamma / \|\boldsymbol{\sigma}^*\|^\gamma$ with $\sigma^0 = (\sigma_1^0, \ldots, \sigma_n^0)^T$.

II. For $k = 1, 2, \ldots,$ until convergence, do:

(i)–(v) as II (i)–(v) in Algorithm 1.

(vi) Apply an iterative method to solve he approximate Jacobian equation

$$J_k\mathbf{c}^{k+1} = \boldsymbol{\sigma}^* - \mathbf{w}^k$$

such that

$$\|J_k\mathbf{c}^{k+1} - \boldsymbol{\sigma}^* + \mathbf{w}^k\| \leq r^k\|\boldsymbol{\sigma}^* - \mathbf{w}^k\|,$$

where

$$r^k = \|\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^*\|^\gamma / \|\boldsymbol{\sigma}^*\|^\gamma, \quad \boldsymbol{\sigma}^k = (\sigma_1^k, \ldots, \sigma_n^k)^T, \quad \sigma_i^k = (\mathbf{u}_i^k)^T A(\mathbf{c}^k)\mathbf{v}_i^k$$

for $1 \leq i \leq n$.

For simplicity, in our numerical experiments we focus on the following four cases: (a) $m = 100$ and $n = 60$; (b) $m = 300$ and $n = 120$; (c) $m = 600$ and $n = 300$; (d) $m = 800$ and $n = 400$. For the ISVP, we first generate the basis matrices $A_0, A_1, \ldots, A_n$ and a solution $\mathbf{c}^*$ randomly. Then we use the singular values of $A(\mathbf{c}^*)$ as the given singular values $\{\sigma_j^*\}_{j=1}^n$. The initial guess $\mathbf{c}^0$ is generated via perturbing each entry of $\mathbf{c}^*$ uniformly distributed in an interval of $[-\max_j|c_j^*| \cdot \beta, \max_j|c_j^*| \cdot \beta]$ for different $\beta > 0$.

For demonstration purposes, all linear systems appearing in Algorithms 1 and 2 were solved by the QMR method [17] via the MATLAB QMR function, where the maximal number of iterations is set to be 1000. In particular, for approximate Jacobian equations in Algorithm 2, we used the preconditioned QMR method with the given stopping tolerance and adopted the MATLAB Incomplete LU factorization as the preconditioner, i.e., LUINC(A, drop-tolerance), where the drop tolerance is set to be 0.01. Also, the initial guess for the approximate Jacobian equation in the $(k + 1)$th outer iteration is set to be $\mathbf{c}^k$ obtained at the kth outer iteration. The stopping tolerance for the other linear systems appearing in Algorithms 1 and 2 is set to be 10^{-14} so that the desired solutions are obtained. Algorithms 1 and 2 stop when

$$\| U_k^T A(\mathbf{c}^k) V_k - \Sigma^* \|_F \le 10^{-8}$$

and the maximal number of outer iterations is set to be 20.

Next, we report our numerical results. Tables 1–4 list the residual values for cases (a)–(d) with different choices of β. Here it. and $\text{cond}_2(J_k)$ denote the number of the kth outer iteration ($k = 0, 1, 2, \dots$) and the condition number of J_k at the kth outer iteration (for Algorithm 2, $\text{cond}_2(J_k)$ means the condition number of J_k for $\gamma = 2$), respectively. For simplicity, in Tables 3 and 4, we only report the numerical results for Algorithm 2 with $\gamma = 2.0$.

From Tables 1–4, we observe that both Algorithms 1 and 2 converge superlinearly in the root sense if approximate Jacobian equations are well-conditioned. However,

Table 1
Residual values for case (a).

		Algorithm 2					Algorithm 1	
β	it.	$\gamma = 1.5$	$\gamma = 1.6$	$\gamma = 1.8$	$\gamma = 2$	$\text{cond}_2(J_k)$		$\text{cond}_2(J_k)$
0.001	0	$8.08e-1$	$8.08e-1$	$8.08e-1$	$8.08e-1$	$3.95e+2$	$8.08e-1$	$3.95e+2$
	1	$2.00e-2$	$1.83e-2$	$1.83e-2$	$1.83e-2$	$3.48e+2$	$1.83e-2$	$3.48e+2$
	2	$6.89e-5$	$5.88e-5$	$5.88e-5$	$5.85e-5$	$3.50e+2$	$8.09e-5$	$3.50e+2$
	3	$1.62e-9$	$1.05e-9$	$1.05e-9$	$1.03e-9$		$6.18e-8$	$3.50e+2$
	4						$2.94e-12$	
0.0001	0	$8.10e-2$	$8.10e-2$	$8.10e-2$	$8.10e-2$	$3.54e+2$	$8.10e-2$	$3.54e+2$
	1	$1.77e-4$	$1.77e-4$	$1.77e-4$	$1.77e-4$	$3.50e+2$	$1.77e-4$	$3.50e+2$
	2	$5.71e-9$	$5.59e-9$	$5.59e-9$	$5.04e-9$		$6.43e-9$	

Table 2
Residual values for case (b).

		Algorithm 2					Algorithm 1	
β	it.	$\gamma = 1.5$	$\gamma = 1.6$	$\gamma = 1.8$	$\gamma = 2$	$\text{cond}_2(J_k)$		$\text{cond}_2(J_k)$
0.001	0	$3.96e+0$	$3.96e+0$	$3.96e+0$	$3.96e+0$	$9.84e+4$	$3.96e+0$	$9.84e+4$
	1	$1.00e-1$	$6.97e-2$	$5.13e-2$	$5.13e-2$	$7.24e+4$	$3.25e-2$	$1.38e+5$
	2	$1.28e-2$	$2.08e-2$	$1.44e-2$	$1.44e-2$	$1.24e+5$	$8.10e-3$	$1.17e+5$
	3	$4.33e-3$	$8.56e-3$	$1.22e-2$	$1.22e-2$	$1.27e+5$	$2.04e-5$	$1.18e+5$
	4	$1.76e-3$	$7.73e-3$	$6.86e-3$	$6.86e-3$	$1.27e+5$	$1.58e-8$	$1.18e+5$
	5	$1.20e-3$	$5.27e-3$	$6.86e-3$	$6.86e-3$	$1.27e+5$	$4.83e-12$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
	20	$5.36e-4$	$9.92e-4$	$6.86e-3$	$6.86e-3$	$1.27e+5$		
0.0001	0	$3.96e-1$	$3.96e-1$	$3.96e-1$	$3.96e-1$	$1.16e+5$	$3.96e-1$	$1.16e+5$
	1	$1.95e-2$	$1.95e-2$	$1.95e-2$	$1.95e-2$	$1.23e+5$	$2.48e-4$	$1.18e+5$
	2	$5.09e-3$	$5.09e-3$	$5.09e-3$	$5.09e-3$	$1.10e+5$	$1.06e-6$	$1.18e+5$
	3	$1.19e-3$	$1.19e-3$	$1.19e-3$	$1.19e-3$	$1.11e+5$	$4.98e-11$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
	20	$8.39e-4$	$8.39e-4$	$8.39e-4$	$8.39e-4$	$1.12e+5$		
0.00001	0	$3.96e-2$	$3.96e-2$	$3.96e-2$	$3.96e-2$	$1.17e+5$	$3.96e-2$	$1.17e+5$
	1	$8.44e-4$	$8.44e-4$	$8.44e-4$	$8.44e-4$	$1.17e+5$	$2.46e-6$	$1.18e+5$
	2	$4.79e-4$	$4.79e-4$	$4.79e-4$	$4.79e-4$	$1.18e+5$	$1.10e-10$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
	20	$4.77e-6$	$4.77e-6$	$4.77e-6$	$4.77e-6$	$1.18e+5$		

TABLE 3
Residual values for case (c).

		Algorithm 2		Algorithm 1	
β	it.	$\gamma = 2$	$\mathrm{cond}_2(J_k)$		$\mathrm{cond}_2(J_k)$
0.0001	0	$3.30e-1$	$2.31e+5$	$3.30e-1$	$2.31e+5$
	1	$1.00e-1$	$2.48e+5$	$2.81e-3$	$2.21e+5$
	2	$1.00e-1$	$2.15e+5$	$4.28e-6$	$2.20e+5$
	3	$1.00e-1$	$1.92e+5$	$9.77e-11$	
	$\vdots$	$\vdots$	$\vdots$		
	20	$1.00e-1$	$1.84e+5$		
0.00001	0	$3.30e-1$	$2.21e+5$	$3.30e-2$	$2.21e+5$
	1	$2.68e-2$	$2.23e+5$	$2.89e-5$	$2.20e+5$
	2	$2.48e-2$	$2.20e+5$	$4.63e-10$	
	$\vdots$	$\vdots$	$\vdots$		
	20	$1.45e-2$	$2.16e+5$		

TABLE 4
Residual values for case (d).

		Algorithm 2		Algorithm 1	
β	it.	$\gamma = 2$	$\mathrm{cond}_2(J_k)$		$\mathrm{cond}_2(J_k)$
0.00001	0	$1.18e-1$	$4.63e+6$	$1.18e-1$	$4.63e+6$
	1	$8.18e-2$	$1.57e+7$	$6.68e-5$	$3.25e+6$
	2	$7.68e-2$	$1.16e+6$	$6.58e-6$	$3.84e+6$
	3	$7.27e-2$	$3.16e+6$	$5.72e-7$	$4.04e+6$
	4	$6.35e-2$	$2.35e+6$	$7.25e-9$	
	$\vdots$	$\vdots$	$\vdots$		
	20	$4.87e-2$	$3.70e+6$		
0.000001	0	$1.18e-2$	$4.11e+6$	$1.18e-2$	$4.11e+6$
	1	$8.15e-3$	$6.00e+6$	$5.57e-7$	$4.05e+6$
	2	$7.44e-3$	$3.60e+6$	$9.06e-10$	
	$\vdots$	$\vdots$	$\vdots$		
	20	$6.26e-3$	$3.55e+6$		

when the condition numbers of approximate Jacobian matrices J_k become large, Algorithm 1 works much better than Algorithm 2.

5. Concluding remarks. In this paper, we propose an Ulm-like method for solving ISVPs. This method avoids solving approximate Jacobian equations in Newton's methods. Under some mild assumptions, we show that our method converges at least quadratically in the root sense. Numerical results illustrate the effectiveness of our method. In our proof, however, we assume that all the given singular values are positive and distinct. An interesting topic is to extend the proposed method to the cases of multiple singular values and of zero singular values, which needs further investigation.

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